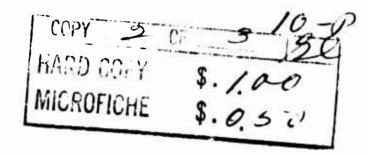
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A NOTE ON THE SOLUTION OF POLYNOMIAL CONGRUENCES

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PREFACE

Part of the Project RAND research program consists of basic supporting studies in mathematics. The present Memorandum makes a contribution to the theory of polynomial congruences.

SUMMARY

As is well known, the number of solutions of the congruence

(1)
$$f(x) = 0(p),$$

where $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$, can be expressed in the form

$$N = \frac{1}{p} \sum_{t,x} e^{2\pi i t f(x)/p},$$

where t and x run independently through the values $0,1,2,\ldots,p-1$. This result is an immediate consequence of the relation

$$\Sigma e^{2\pi i t y/p} = 0, y \neq 0(p),$$

t = p, y = 0(p).

In this note we present an alternative expression for the number of solutions of (1).

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A NOTE ON THE SOLUTION OF POLYNOMIAL CONGRUENCES

1. INTRODUCTION

It is well known that the number of solutions of the congruence

(1.1)
$$f(x) = 0(p)$$
,

where $f(x) = x^n + a_1 x^{n-1} + \cdots + a_n$; can be expressed in the form

(1.2)
$$N = \frac{1}{p} \sum_{t,x} e^{2\pi i t f(x)/p},$$

where t and x run independently through the values $0,1,2,\ldots,p-1$. This result is an immediate consequence of the relation

In this note we present an alternative expression for the number of solutions of (1.1).

2. AN EQUIVALENT VECTOR-MATRIX CONGRUENCE

The equation f(x) = 0 is readily seen to be the characteristic equation of the matrix

(2.1)
$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & & & & & \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix};$$

see [1], p. 225.

Using arguments completely analogous to that for the complex field, we see that a necessary and sufficient condition for a nontrivial solution of the vector-matrix congruence

(2.2) Ax =
$$\lambda x(p)$$
,

where x is now the n-dimensional column vector with components x_1, x_2, \dots, x_n , and λ is a scalar, is that

$$(2.3) f(\lambda) = 0(p).$$

Each root of (2.3) generate a ray of solutions kx, where k = 1, 2, ..., p - 1.

3. MULTIDIMENSIONAL EXPONENTIAL SUM

Let t be an n-dimensional vector with components t_1, t_2, \dots, t_n , and let (t,x) denote, as usual, the vector inner product. We can then write, as the number of nontrivial solutions of (2.2),

(3.1)
$$\sum_{\substack{\Sigma \ \Sigma \ \Sigma' \ e}} \frac{2\pi i}{p} (t, Ax - \lambda x),$$

where (u,v) denotes the usual inner product and the prime denotes the fact that x = 0 is omitted in the summation.

Since each solution of $f(\lambda) = 0$ generates p-1 solutions of (2.2), we have

(3.2)
$$N = \frac{1}{p^{n}(p-1)} \sum_{t} \sum_{\lambda} \sum_{z}' e^{\frac{2\pi i}{p}(t,Ax-\lambda x)}.$$

We can eliminate the prime by writing

(3.3)
$$N = \frac{1}{p^{n}(p-1)} \sum_{t,\lambda,x} e^{\frac{2\pi i}{p}(t,Ax-\lambda x)} - \frac{p}{p-1}.$$

Summing over the scalar λ first, we have finally

(3.4)
$$N = \frac{1}{p^{n-1}(p-1)} \sum_{(t,x)=0(p)} e^{2\pi i (t,Ax)/p} - \frac{p}{p-1}.$$

If A is symmetric, we write t = u + v, x = u - v, and obtain

(3.5)
$$N = \frac{1}{p^{n-1}(p-1)} \sum_{(u,u)=(v,v)(p)} e^{\frac{2\pi i}{p}[(u,Au)-(v,Av)]} - \frac{1}{p-1}.$$

This, in turn, may be written

(3.6)
$$N = \frac{1}{p^{(n-1)}(p-1)} \sum_{k | (u,u)=k(p)} \frac{2\pi i}{p} (u,Au) |^{2}$$
$$-\frac{p}{p-1},$$

an interesting formula.

4. EXAMPLE

Consider the congruence

(4.1)
$$\lambda^3 + a = 0(p)$$
.

The corresponding matrix is

(4.2)
$$A = \left[\begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a & 0 & 0 \end{array} \right].$$

Hence, the number of solutions of (4.1) is given by

(4.3)
$$N = \frac{1}{p^2(p-1)} \sum_{S} e^{\frac{2\pi i}{p}(t_1 x_2 + t_2 x_3 - at_3 x_1)} - \frac{p}{p-1},$$

where the set of values S is determined by $t_1x_1 + t_2x_2 + t_3x_3 = 0$.

REFERENCE

1. Bellman, R., <u>Introduction to Matrix Analysis</u>, McGraw-Hill Book Company, Inc., New York, 1960.